Sequences - Classwork

In mathematics, we define a sequence as a set of numbers that has an identified first member, second member, third member, etc. It is a function whose domain in the set of positive integers. But rather than use standard function notation, we use subscript notation because it better defines the sequence.

$$\begin{array}{cccccc} 1, & 2, & 3, & 4, & \ldots & n, & \ldots \\ | & | & | & | & & | & \\ a_1, & a_2, & a_3, & a_4, & & a_n, & \ldots \end{array}$$

The first term in the sequence is a_1, the 2nd is a_2,the nth is a_n

The entire sequence is denoted by $\{a_n\}$.

Example 1) Write the terms of the sequence $\{a_n\} = \{4 + (-1)^n\}$

Example 2) Write the terms of the sequence $\{b_n\} = \left\{\dfrac{3n}{n+1}\right\}$

Example 3) Write the terms of the sequence $\{c_n\} = \left\{\dfrac{n^2-1}{2^n-1}\right\}$.

We are mostly interested in sequences whose terms approach a limiting value. Such sequences are said to **converge**. The sequence $\left\{\dfrac{1}{2^n}\right\} = \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dfrac{1}{32}, \dfrac{1}{64}, \ldots$ converges to zero. We say that $\lim\limits_{n\to\infty} a_n = 0$. If a sequence has no limit (gets infinitely big or small or oscillates between values), the sequence is said to **diverge**.

We are making a comparison to functions that have a limit as opposed to sequences having a limit. If f is a function such that $\lim\limits_{x\to\infty} f(x) = L$ and $\{a_n\}$ is a sequence such that $f(n) = \{a_n\}$ for every positive integer n, then $\lim\limits_{n\to\infty} a_n = L$.

Example 4) Find $\lim\limits_{n\to\infty} a_n$ if $a_n = \left(1 - \dfrac{1}{n}\right)^n$.

We know that by L'Hopital's rule, $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x =$ ________________

So, $\lim\limits_{n\to\infty} a_n =$ ________________

Example 5) Determine whether the sequence $\{a_n\} = \{5 + (-1)^n\}$ converges or diverges.

Example 6) Determine whether the sequence $\{b_n\} = \left\{\dfrac{n}{1-50n}\right\}$ converges or diverges.

Example 7) Show that the sequence whose *n*th term is $a_n = \dfrac{n^2}{3^n - 3}$ converges.

The Squeeze Theorem for Sequences
If both $\lim\limits_{x\to\infty} a_n = L$ and $\lim\limits_{x\to\infty} b_n = L$, and there exists an integer *n* such that $a_n \le c_n \le b_n$ for all $n > N$, then $\lim\limits_{x\to\infty} c_n = L$ as well.

Example 8) Show that the sequence $\left\{(-1)^n \dfrac{1}{n!}\right\}$ converges and find its limit.

You can make a chart to examine the function: It appears that the limit is ________________

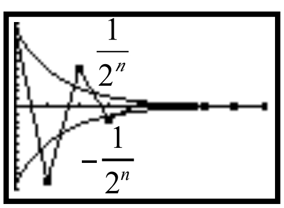
n	0	1	2	3	4	5	6
$\left\{(-1)^n \frac{1}{n!}\right\}$							

But this is not a proof. To use the squeeze theorem, we need two convergent sequences. Let's examine $a_n = \dfrac{1}{3^n}$ and $b_n = -\dfrac{1}{3^n}$ both of which converge to zero. Again, complete the chart for

n	0	1	2	3	4	5	6
$n!$							
3^n							

What you can now see that if $n \ge 4$ then $2^n < n!$. That means that $-\dfrac{1}{2^n} \le (-1)^n \dfrac{1}{n!} \le \dfrac{1}{2^n}$ for $n > 4$

By the Squeeze theorem, $\lim\limits_{x\to\infty} (-1)^n \dfrac{1}{n!} = 0$



Note how $-\dfrac{1}{2^n} \le (-1)^n \dfrac{1}{n!} \le \dfrac{1}{2^n}$ for $n > 4$. Also realize that $(-1)^n \dfrac{1}{n!}$ is only defined for integer values of n.

This example is instructive because it shows that although $\lim\limits_{x\to\infty} 3^n = \infty$ and $\lim\limits_{x\to\infty} n! = \infty$, $n!$ grows so much faster than 3^n and hence $\lim\limits_{x\to\infty} \dfrac{3^n}{n!} = 0$. In fact, it can be shown that $\lim\limits_{x\to\infty} \dfrac{k^n}{n!} = 0$.

Although this course does not require you to be able to generate the *n*th term of a sequence if you are given the terms of the sequence, it is kind of fun to do so. Can you find the *n*th term of the following sequences? a. through d. are relatively simple and e. and f. are harder.

a. $\frac{2}{1},\frac{4}{3},\frac{8}{5},\frac{16}{7},\frac{32}{9},\ldots$

b. $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\ldots$

c. $\frac{1}{2},\frac{\sqrt{2}}{3},\frac{\sqrt[3]{3}}{4},\frac{\sqrt[4]{4}}{5},\ldots$

d. $1,0,-1,0,1,0,-1,\ldots$

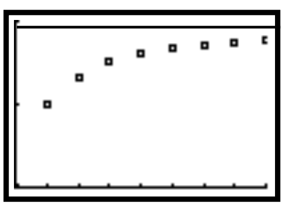
e. $\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{15},\ldots$

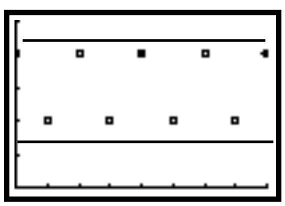
f. $-\frac{2}{1},\frac{8}{2},-\frac{26}{6},\frac{80}{24},-\frac{242}{120},\ldots$

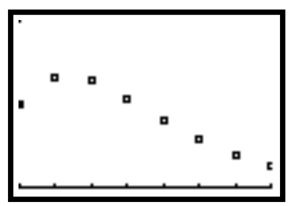
Finally, we define a **monotonic** sequence as a sequence whose terms get successively larger or smaller.

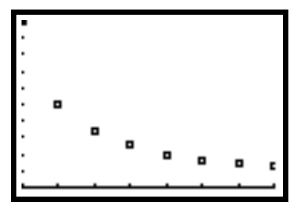
A sequence $\{a_n\}$ is monotonic if its terms are nondecreasing: $a_1 \le a_2 \le a_3 \le a_4 \le \ \ldots \ \le a_n \le \ldots$ or
A sequence $\{a_n\}$ is monotonic if its terms are nonincreasing: $a_1 \ge a_2 \ge a_3 \ge a_4 \ge \ \ldots \ \ge a_n \ge \ldots$

Monotonic curves are bounded either above or below (or both) by some number *M*. If a sequence is bounded and monotonic, then it converges.


monotonoic and bounded


Not monotonic but bounded


Not monotonic (2nd term) but bounded


monotonic and bounded

Sequences - Homework

Write the first 5 terms of the sequence.

1. $a_n = \frac{2n}{n+2}$

2. $a_n = \cos n\pi$

3. $a_n = (-1)^n\left(2 - \frac{1}{n} + \frac{1}{n^2}\right)$

4. $a_n = \frac{(2n)!}{(n+1)!}$

Write an expression for the *n*th term of the sequence

5. $1,5,9,13,\ldots$

6. $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},\ldots$

7. $2,-1,\frac{1}{2},-\frac{1}{4},\frac{1}{8},\ldots$

8. $-1,2,7,14,23$

In the following sequences, determine the convergence or divergence with the given *n*th term

9. $a_n = (-1)^n \left(\dfrac{n}{n+1}\right)$

10. $a_n = 5 + (-1)^n$

11. $a_n = \dfrac{3n^2 - 3n + 8}{4n^2 + 1}$

12. $a_n = \dfrac{2\sqrt{n}}{3\sqrt{n} + 1}$

13. $a_n = \dfrac{\ln(n^3)}{n}$

14. $a_n = \dfrac{4^n}{5^n}$

15. $a_n = \dfrac{(n+2)\,!}{(n+1)!}$

16. $a_n = \dfrac{(n-2)\,!}{(n+1)\,!}$

17. $a_n = n \sin \dfrac{1}{n}$

In the following exercises, use your grapher to determine whether the sequence is monotonic and if it is bounded.

18. $a_n = 5 - \dfrac{1}{n}$

19. $a_n = \dfrac{\sin n + \cos n}{n}$

20. $a_n = ne^{-n/2}$

21. $a_n = \left(\dfrac{4}{5}\right)^n$

22. $a_n = \left(\dfrac{5}{4}\right)^n$

23. $a_n = \dfrac{n}{3^{n+1}}$