Infinite Series Review Problems
AP Calculus BC

Name: ANSWERS

For Questions 1-5, determine whether each series converges or diverges. Justify your answer.

(1) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

COMPARE $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES HARMONIC

$\lim_{n\to\infty} \frac{n^2}{n^3+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$ POSITIVE FINITE

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ ALSO DIVERGES BY LCT.

(2) $\sum_{n=1}^{\infty} \frac{3n^3-n}{1-2n^3}$

$\lim_{n\to\infty} \frac{3n^3-n}{1-2n^3} = 3/2 \neq 0$

$\therefore \sum_{n=1}^{\infty} \frac{3n^3-n}{1-2n^3}$ DIVERGES BY nth TERM TEST.

(3) $\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$

COMPARE $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES HARMONIC SERIES.

$\frac{1}{n-\ln n} \geq \frac{1}{n}$ $\quad n \geq n-\ln(n)$ ✓

SINCE $\frac{1}{n-\ln(n)} \geq \frac{1}{n}$, BOTH SERIES DIVERGE BY DCT.

(4) $\sum_{n=0}^{\infty} \frac{2^{2n}}{n!}$

$\lim_{n\to\infty} \left| \frac{2^{2n+2}}{(n+1)!} \cdot \frac{n!}{2^{2n}} \right|$

$\lim_{n\to\infty} \left| \frac{2^2}{(n+1)} \right| = 0 < 1$

$\therefore \sum_{n=0}^{\infty} \frac{2^{2n}}{n!}$ CONVERGES BY RATIO TEST.

(5) $\sum_{n=1}^{\infty} \left(\frac{3}{4n}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3}{4n}\right)^n\right|}$

$\lim_{n\to\infty} \frac{3}{4n} = 0 < 1$

$\therefore \sum_{n=1}^{\infty} \left(\frac{3}{4n}\right)^n$ CONVERGES BY ROOT TEST.

(6) For what integer k, $k > 1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{4}\right)^n$ converge?

(A) 6

(B) 5

(C) 4

$\boxed{\text{(D) 3}}$

(E) 2

(7) Find the exact sum of $\sum_{n=0}^{\infty}\left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)}\right]$

$$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n \qquad - \qquad \sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)} = \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$S = \frac{1}{1-2/3} = 3 \qquad\qquad = \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \cdots$$

$$= 1$$

$$\sum_{n=0}^{\infty}\left[\left(\frac{2}{3}\right)^n - \frac{1}{(n+1)(n+2)}\right] = 3 - 1 = \boxed{2}$$

(8) Which of the following series converge?

I. $\sum_{n=1}^{\infty}\frac{n}{n+2}$ II. $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}$ III. $\sum_{n=1}^{\infty}\frac{1}{n}$

(A) None $\boxed{\text{(B) II only}}$ (C) III only (D) II and III only (E) I, II, and III

(9) Determine whether the series given below converges or diverges. If the series converges, define whether the series conditionally converges or absolutely converges. Justify your answer.

$$\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n+1}$$

1) $\lim \frac{\sqrt{n}}{n+1} = 0$

2) DECREASING $\frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1}$

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n+1}$ CONVERGES BY AST

CHECK:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n\sqrt{n}}{n+1}\right| = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$$

COMPARE $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

DIVERGES p-SERIES b/c $p = 1/2 < 1$.

$\lim_{n\to\infty}\frac{\sqrt{n}}{n+1}\cdot\frac{\sqrt{n}}{1} = 1$ (POSITIVE, FINITE #)

$\therefore \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+1}$ ALSO DIVERGES BY LCT

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^n\sqrt{n}}{n+1}$ CONVERGES CONDITIONALLY

For Questions 10-12, determine whether each series converges or diverges. Justify your answer. If the series converges, find the exact summation or an interval of approximation using S_4.

(10) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$

$$\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}}dx$$

$$\lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^{3/2}}dx$$

$$\lim_{b\to\infty} \left[\frac{-2}{\sqrt{\ln x}}\right]\Big|_2^b = \lim_{b\to\infty}\left[\frac{-2}{\sqrt{\ln b}} + \frac{2}{\sqrt{\ln 2}}\right] = \frac{2}{\sqrt{\ln 2}}$$

SINCE $\frac{1}{n(\ln n)^{3/2}}$ IS POSITIVE, CONTINUOUS, & DECREASING AND $\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}}$ CONVERGES, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ ALSO CONVERGES BY $\int$ TEST.

$S_4 = 1.46701$

$$R_4 \le \int_5^{\infty} \frac{-2}{\sqrt{\ln x}} = \frac{2}{\sqrt{\ln 5}}$$

$$\boxed{1.4670 \le S \le 2.9835}$$

(11) $\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{243}$

$r = 2/3$

$$S = \frac{4/9}{1-2/3} = \frac{4/9}{1/3} = \boxed{\frac{4}{3}}$$

(12) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}}$

1) $\lim_{n\to\infty} \frac{1}{n^{2/3}} = 0$

2) $\frac{1}{(n+1)^{2/3}} \le \frac{1}{n^{2/3}}$ DECREASING

$\therefore \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}}$

$S_4 = -.4539$

$$R_4 \le \left|\frac{1}{5^{2/3}}\right| = 0.3419$$

$$\boxed{-.7959 \le S \le -.1119}$$

(13) Find the smallest sum that will be accurate within 0.003 of the actual sum of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^4+5}$.

$\boxed{S_6 = 0.0943}$

(14) If $\lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$ is finite, then which of the following must be true?

$\boxed{\text{(A) } \sum_{n=1}^{\infty}\frac{1}{n^p} \text{ converges}}$

(B) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges

(C) $\sum_{n=1}^{\infty}\frac{1}{n^{p-2}}$ converges

(D) $\sum_{n=1}^{\infty}\frac{1}{n^{p-1}}$ converges

(E) $\sum_{n=1}^{\infty}\frac{1}{n^{p+1}}$ diverges

(15) Using the Integral Test, prove whether the following converges or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} ne^{-n} \qquad \int_1^{\infty} xe^{-x}\,dx$$

$$\lim_{b\to\infty}\int_1^b xe^{-x}\,dx$$

$$\lim_{b\to\infty}\left[\frac{-x}{e^x} - \frac{1}{e^x}\right]\Bigg|_1^b = \left(\frac{-b}{e^b} - \frac{1}{e^b}\right) - \left(-\frac{1}{e} - \frac{1}{e}\right) = \frac{2}{e}$$

SINCE ne^{-n} IS POSITIVE, CONTINUOUS, & DECREASING

AND $\int_1^{\infty} xe^{-x}\,dx$ CONVERGED, THEN $\sum_{n=1}^{\infty} ne^{-n}$ ALSO

CONVERGES BY INTEGRAL TEST.