WARM UP - Calc Active

1. Approximate the sum of the converging alternating series using the first six terms.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,2}{n^3}$$

2. Find the sum:

$$1-\frac{100}{2!}+\frac{10{,}000}{4!}+\cdots+\frac{(-1)^n \times 10^{2n}}{(2n)!}+\cdots$$

Lagrange Error Bound

Objective:

- Find and use the remainder of a Taylor polynomial (Lagrange error)
- Solve FRQs involving series.

Lagrange Form of the Remainder

aka Lagrange Error Bound
aka Taylor's Theorem Remainder

To measure the accuracy of approximating a function with a Taylor Polynomial we can use the following formula:

$$f(x) = P_n(x) + R_n(x)$$

exact value — approximate value — remainder

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exact value — approximate value — remainder

so $R_n(x) = f(x) - P_n(x)$

To find the error of this approximation we take the absolute value of this.

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|$$

THEOREM 9.19 Taylor's Theorem

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}. \right|$$

Lagrange Form of the Remainder

Note: For all of the following examples, the function f has derivatives of all orders for all real numbers.

Ex.1: Calc Active

Assume that f(2) = 6, f'(2) = 4, f''(2) = -7, f'''(2) = 8.

(a) Write the third-degree Taylor polynomial for f about x = 2, and use it to approximate f(2.3).

(b) The fourth derivative of f satisifies the inequality $|f^{(4)}(x)| < 9$ for all x in the closed interval [2, 2.3]. Find the error bound of the approximation.

Ex. 2: Calc Inactive

a)Find the fifth-degree Maclaurin polynomial for sinx and approximate sin1using the polynomial. Show that:

$$|P_5(1) - f(1)| < \frac{1}{700}$$

Ex. 3: Calc Active

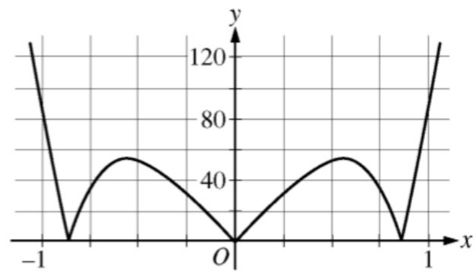
The function f has $f^{(4)}(x)=e^{\sin x}$. If a third-degree Taylor polynomial for f about x = 0 is used on the interval [0,1], what is the Lagrange error bound for the error on the interval of [0,1]?

Ex. 4: Calc Inactive

The Taylor series about x=3 for a certain function f converges to f(x) for all x in the IOC. The nth derivative of f at x=3 is:

$$f^{(n)}(3)=\frac{(-1)^n n!}{5^n(n+3)} \text{ and } f(3)=\frac{1}{3}$$

Show that the third-degree Taylor polynomial approximates f(4) with an error less than 1/4000.



Graph of $y = \left|f^{(5)}(x)\right|$

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.