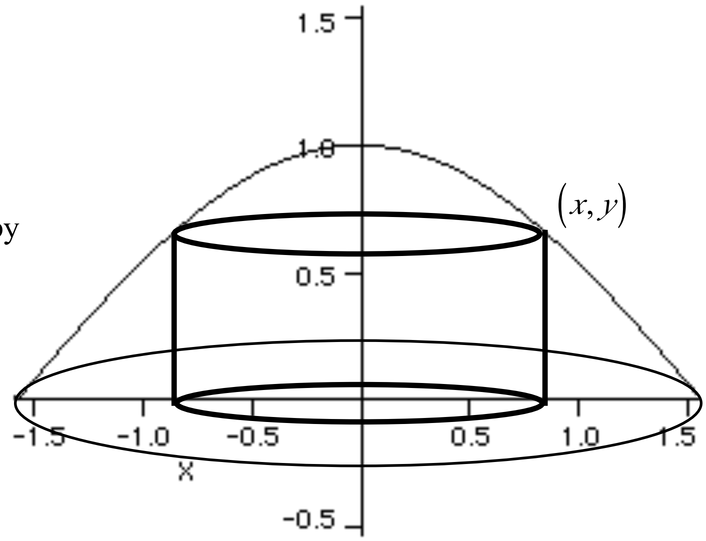
Integration by Parts - Classwork

Suppose you wish to rotate about the y-axis the graph of $y = \cos x$

The volume, using the shell method is given by

$$V = 2\pi\int_0^{\pi/2} x \cdot \cos x \, dx$$



We can obviously solve this using your calculator, which uses the techniques of numerical integration (trapezoids, Simpson's, etc.) But *u*-substitution for the Fundamental Theorem of Calculus doesn't work here, and to this point in your study of calculus, you were at a dead end. We will now learn techniques to solve such problems that were essential in the days "BC" (before calculators). These techniques are now interesting for historical reasons, but are important for this course.

Let us start by learning how to integrate a product of two functions. We know how to *differentiate* a product, but how about an integral? Let us start for the formula for the differential of a product. If $y = u \cdot v$ where *u* and *v* are differentiable functions of *x*, then

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$	If we multiply both sides by *dx*, we get
$dy = u \cdot dv + v \cdot du$	If we integrate both sides of the equation, we get
$\int dy = \int u \cdot dv + \int v \cdot du$	Let's move some terms around
$\int u \cdot dv = \int dy - \int v \cdot du$	Since $\int dy = y$ (ignoring the +C), we get
$\int u \cdot dv = y - \int v \cdot du$	But since, $y = uv$ we get, finally
$\int u \cdot dv = uv - \int v \cdot du$	This is the formula for what we call integration by parts

Technique: Integration by Parts

If *u* and *v* are differentiable functions of *x,* then

$$\int u \cdot dv = uv - \int v \cdot du$$

Technique: Integration by Parts

If u and v are differentiable functions of x, then

$$\int u \cdot dv = uv - \int v \cdot du$$

Example 1) Find $\int x \cdot \cos x \, dx$ To do this by integration by parts, set up a chart

$u =$ ________ $v =$ ________
$du =$ ________ $dv =$ ________

There are three pieces of information that must be placed into the u and dv slots: x, cos x, and dx
dx will always go into the dv slot. Let's put the x in the u slot, and cos x in the dv slot. So we have.

$u = x$ $v = \sin x$
$du = dx$ $dv = \cos x \ dx$ We can now complete the chart

$u = x$ $v = \sin x$
$du = dx$ $dv = \cos x \ dx$

The integration by parts formula works like this:

$u = x \longrightarrow v = \sin x$
$du = dx \longleftarrow dv = \cos x \ dx$

$\int x \cdot \cos x \, dx = x \cdot \sin x - \int \sin x \, dx$ What we have done essentially is create another integral. which hopefully we can take.

$= x \cdot \sin x + \cos x + C$

Let's check it out: take the derivative of $= x \cdot \sin x + \cos x + C$

Example 2) Find $\int x \cdot e^x dx$ and $\int_0^1 xe^x dx$.

First, make sure that normal u-substitution does not work before you try "Parts".

$u =$ ________ $v =$ ________
$du =$ ________ $dv =$ ________

Example 3) Find $\int x^2 \ln x \, dx$

$u =$ ________ $v =$ ________
$du =$ ________ $dv =$ ________

Example 4) Find $\int \ln x \, dx$

$u = e^x \qquad v = \_\_\_\_\_\_\_\_$

$du = \_\_\_\_\_\_\_ \qquad dv = x\,dx$

When things go wrong: Integration by parts trades one integral for another. The bargain is worth making only if the second expression is easier to integrate than the original. For instance, in example 2 above, $\int x \cdot e^x dx,$ suppose you set up your chart like this:

$u = e^x \qquad v = \_\_\_\_\_\_\_\_$

$du = \_\_\_\_\_\_\_ \qquad dv = x\,dx$

Complete the chart.

Now set up integration by parts. Does it help?

Our choices aren't wrong, they just don't help. This time we struck out.

Try taking $\int 2x\, e^{x^2} dx$by integration by parts. There are two possibilities. Try each.

$u = \_\_\_\_\_\_\_\_ \qquad v = \_\_\_\_\_\_\_\_ \qquad\qquad u = \_\_\_\_\_\_\_\_ \qquad v = \_\_\_\_\_\_\_\_$

$du = \_\_\_\_\_\_\_ \qquad dv = \_\_\_\_\_\_\_\_ \qquad\qquad du = \_\_\_\_\_\_\_ \qquad dv = \_\_\_\_\_\_\_\_$

Ths conclusions is that integration by parts is the wrong tool for the job. This problem can be done by ordinary *u*-substitution. Try it.

When things go right: The trick is to choose *u* and *v* successfully. In practice, "successfully" means two things: *dv* can be antidifferentiated to give *u*, and $\int v \cdot du$ is simpler than $\int u \cdot dv$.

Example 5) Find $\int \arctan x \, dx$

$u = \_\_\_\_\_\_\_\_ \qquad v = \_\_\_\_\_\_\_\_$

$du = \_\_\_\_\_\_\_ \qquad dv = \_\_\_\_\_\_\_\_$

Repeated Integration by Parts: Sometimes, we can repeat the process of integration by parts to integrate a difficult expression.

Example 6) Find $\int x^2 e^x dx$ Set up your chart:

$u = \_\_\_\_\_\_\_\_ \qquad v = \_\_\_\_\_\_\_\_$

$du = \_\_\_\_\_\_\_ \qquad dv = \_\_\_\_\_\_\_\_$

So $\int x^2 e^x dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$ Did we get anywhere?

What is the final answer?

Example 7) Find $\int e^x \sin x \, dx$. *u*-substitution doesn't work. Try parts.

$u =$ ________ $v =$ ________
$du =$ ________ $dv =$ ________

So $\int e^x \sin x \, dx =$ ____________. We still have a tough integral. Parts again.

$u =$ ________ $v =$ ________
$du =$ ________ $dv =$ ________

So $\int e^x \sin x \, dx =$ __________________________. The original integral reappeared. Are we chasing our tails?

Example 8) Tougher one. $\int \sin^2 x \, dx$

Example 9) Tougher one. $\int x^2 \cos 4x \cdot dx$

Integration by Parts - Homework

1. $\int x \sin x \, dx$

2. $\int x \cos(5x) \, dx$

3. $\int xe^{8x} \, dx$

4. $\int 6xe^{-3x} \, dx$

5. $\int x \sec^2 x \, dx$

6. $\int \sqrt{x} \ln x \, dx$

7. $\int x\sqrt{1+x} \, dx$

8. $\int \sin^{-1} x \, dx$

9. $\int (x+4)\, e^{2x} dx$

10. $\int x^3 \ln x \, dx$

11. $\int x^3 e^x \, dx$

12. $\int x^2 \sin x \, dx$

13. $\int x^2 \cos x \, dx$

14. $\int e^x \cos x \, dx$

15. $\int_1^e 4x \ln x \, dx$

16. Find the volume when $y = \ln x$ is rotated about the x - axis between $x = 1$ and $x = e$.